THE PERIOD MATRIX OF THE HYPERELLIPTIC CURVE $w^2 = z^{2g+1} - 1$

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ABSTRACT. We introduce a geometric algorithm for finding a symplectic basis of the first integral homology group of a compact Riemann surface which is a p-cyclic covering of $\mathbb{C}P^1$ branched over 3 points. It gives us an unknown symplectic basis of the hyperelliptic curve defined by the affine equation $w^2 = z^{2g+1} - 1$ for genus $g \geq 2$. Moreover, we explicitly obtain the period matrix of this curve whose all entries are elements of the $(2g+1)$-st cyclotomic field.

1. INTRODUCTION

Let X be a compact Riemann surface of genus $g \geq 2$ or smooth projective algebraic curve over $\mathbb{C}$. The period matrix τ_g of X depends only on the choice of a symplectic basis of the first integral homology group $H_1(X;\mathbb{Z})$. It is known that τ_g is symmetric and its imaginary part is positive definite. The Jacobian variety $J(X)$ of X is defined by a complex torus $\mathbb{C}^g/(\mathbb{Z}^g + \tau_g\mathbb{Z}^g)$. Torelli theorem says that two given Riemann surfaces X and Y are biholomorphic if and only if $J(X)$ and $J(Y)$ are isomorphic as polarized abelian varieties. It implies τ_g determines the complex structure of X. In general, it is not easy to calculate τ_g, in particular, find a symplectic basis of $H_1(X;\mathbb{Z})$. Tretkoff and Tretkoff [17] gave a method of computation of τ_g, using *Hurwitz systems*. Bene [3] showed a way of finding a symplectic basis of $H_1(X;\mathbb{Z})$ by means of *chord slides* for linear chord diagrams. By combining these two methods, we explicitly write down a geometric algorithm for finding a symplectic basis of the first integral homology groups of p-cyclic coverings of $\mathbb{C}P^1$ branched over 3 points for prime number $p \geq 5$. Furthermore we explain calculating the period matrix this kind of Riemann surfaces. We compute the period matrices of the hyperelliptic and Klein quartic curve defined by the affine equations $y^7 = x(1-x)$ and $y^7 = x(1-x)^2$ respectively.

For generic genus, few examples of period matrices are known. Schindler [13] computed the period matrices of three types of hyperelliptic curves of genus $g \geq 2$. These are the unique examples as far as we know. Kamata [10] introduced an algorithm for calculating those of Fermat type curves. For low genus case, see [6], [7], and their references. We remark that Streit [14] studied about the period matrices from viewpoints of representation theory. For any *odd* number $q \geq 5$, let $C_{q,1}$ be the smooth projective curve over $\mathbb{C}$ defined by the affine equation $y^q = x(1-x)$. This is biholomorphic to the hyperelliptic curve defined by the affine equation $w^2 = z^{2g+1} - 1$ of genus $g = (q-1)/2$. Tashiro, Yamazaki, Ito, and Higuchi [16] computed the periods on $C_{q,1}$. We obtain the period matrix τ_g of $C_{q,1}$ using the inverse of the Vandermonde matrix. Schindler [13] obtained one of the same hyperelliptic curve of genus $g \geq 2$ defined by the affine equation $w_1^2 = z_1(z_1^{2g+1} - 1)$. This result contains a recurrence relation. On the other side, we have an explicit representation of it. Set $\zeta = \zeta_q = \exp(2\pi\sqrt{-1}/q)$. A symplectic

basis $\{A_i, B_i\}_{i=1,2,\dots,g}$ of $H_1(C_{q,1};\mathbb{Z})$ is defined later. For variables $x_1, x_2, \dots, x_n$, we denote by $\sigma_i(x_1, x_2, \dots, x_n)$ the symmetric polynomial

$$\sum_{1\le j_1<\cdots<j_i\le n} x_{j_1}\cdots x_{j_i}$$

for $1 \le i \le n$ and $\sigma_0(x_1, x_2, \dots, x_n) = 1$.

Theorem 4.5. *We have the period matrix τ_g of $C_{q,1}$ with respect to the symplectic basis $\{A_i, B_i\}_{i=1,2,\dots,g}$*

$$\tau_g = \left(\sum_{k=1}^{g} \frac{(-1)^{i+g}}{2g+1}(1-\zeta^{2kj})\sigma_{g-i}(\zeta^2, \zeta^4, \dots, \widehat{\zeta^{2j}}, \dots, \zeta^{2g}) \prod_{m=g-k+1}^{2g-k}(1-\zeta^{2m}) \right)_{i,j},$$

where the 'hat' symbol $\widehat{\ }$ over ζ^{2j} indicates that this element is deleted from the sequence $\zeta^2, \zeta^4, \dots, \zeta^{2g}$.

Contents

2. Algorithm

In this section, we write down a geometric algorithm for finding a symplectic basis of the first integral homology groups of smooth projective algebraic curves defined by the affine equation $y^p = x^l(1-x)^m$. Here $p \ge 5$ is prime number, l, m are coprime, and $1 \le l, m, l+m < p$. We denote this curve by $X_{p,l,m}$. This is a Riemann surface of genus $g = (p-1)/2$ and can be considered as a p-sheeted cyclic coverings of $\mathbb{C}P^1$ branched over $\{0, 1, \infty\} \subset \mathbb{C}P^1$. In particular, we simply write the curve $X_{p,1,m} = C_{p,m}$ for the case $l = 1$. Through this section, we use an example: the Klein quartic $C_{7,2}$. Moreover, we explain calculating the period matrix of $X_{p,l,m}$ using holomorphic 1-forms of Bennama and Carbonne [5]. It is known that there are only two $X_{7,l,m}$ up to isomorphism for

the case $p = 7$. They are $C_{7,1}$ and $C_{7,2}$. See [8, §1.3.2] for example. We calculate the period matrices of them.

2.1. **Dessins d'enfants.** Let X be a smooth projective algebraic curve over a field k. We assume that k is $\mathbb{C}$ and there exists a covering $\pi\colon X \to \mathbb{C}P^1$ branched over $\{0, 1, \infty\} \subset \mathbb{C}P^1$. The inverse image $\pi^{-1}([0, 1])$ in X of the unit interval in $\mathbb{C}P^1$ is called dessins d'enfants [9]. It is a topological bipartite graph illustrated on the Riemann surface X. Belyi [2] proved that all the algebraic curves over $\overline{\mathbb{Q}}$ correspond to the dessins d'enfants. The map π is often called Belyi map. See also [18]. In the rest of this paper, we simply assume that X is $X_{p,l,m}$ and $\pi\colon X \ni (x, y) \mapsto x \in \mathbb{C}P^1$ is a p-cyclic covering branched over $\{0, 1, \infty\} \subset \mathbb{C}P^1$. Set the order p holomorphic automorphism $\sigma(x, y) = (x, \zeta_p y)$. Here, we denote $\zeta_p = \exp(2\pi\sqrt{-1}/p)$. Let $y_0(t)$ is a real analytic function $\sqrt[p]{t^l(1-t)^m}$. A continuous path $I_0\colon [0, 1] \to X$ is defined by the equation $I_0(t) = (t, y_0(t)) \in X$ for $0 \le t \le 1$. Immediately we obtain $\pi(I_0) = [0, 1] \subset \mathbb{C}P^1$ and the dessin d'enfants $\pi^{-1}([0, 1]) = \cup_{i=0}^{p-1} \sigma_*^i(I_0)$. We call $\pi^{-1}(0)$ and $\pi^{-1}(1)$ the white and black vertex respectively. The dessin $\pi^{-1}([0, 1])$ in X is a bipartite graph. Take a point b_i on the $\sigma_*^i(I_0)$ except for endpoints for each i. For the Klein quartic $C_{7,2}$, We draw the dessin d'enfants $\pi^{-1}([0, 1])$. See Figure 1.

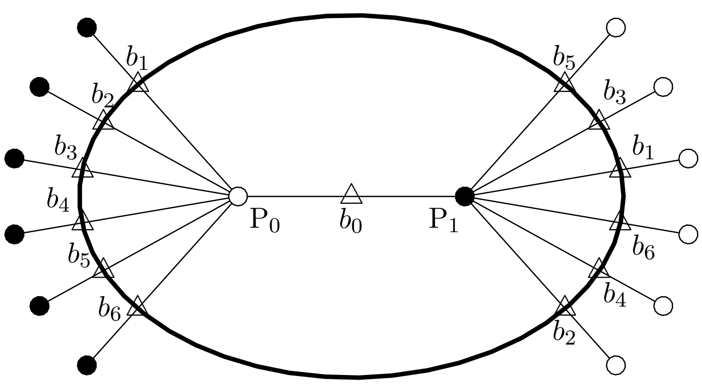

FIGURE 1. A dessin d'enfants of the Klein quartic $C_{7,2}$

2.2. **Intersection numbers.** We introduce the method of Tretkoff and Tretkoff [17], called Hurwitz system, from dessins d'enfants to the intersection numbers of the loops in X. First, we explain the definition of a chord diagram. A chord diagram is defined by a union of a unit circle S^1 and $2g = p - 1$ simple chords. It satisfies the endpoints of the chords attached to S^1 and each arc intersects at most one points. If the chord diagrams are oriented, we call it an oriented chord diagram.

For $i = 1, 2, \ldots, p-1$, let c_i denote the loop $I_0 \cdot \sigma_*^i(I_0)^{-1}$ in X. Here, the product $I_0 \cdot \sigma_*^i(I_0)^{-1}$ indicates that we traverse I_0 first, then $\sigma_*^i(I_0)^{-1}$. It follows that the dessin d'enfants $\pi^{-1}([0, 1])$ equals to the union $\cup_{i=0}^{p-1} c_i$. We deform the dessin topologically and consider c_i as not loops but chords. We get an oriented chord diagram and compute the intersection numbers $c_i \cdot c_j = 0$ or ± 1. Let $(a_{i,j})_{i,j}$ denote the matrix whose (i, j)-th

entry is $a_{i,j}$. If the $2g \times 2g$ intersection matrix $(c_i \cdot c_j)_{i,j}$ is regular, then $\{c_i\}_{i=1,2,\dots,2g}$ is a basis of the first integral homology group $H_1(X;\mathbb{Z})$.

For $C_{7,2}$, we obtain the oriented chord diagram in Figure 2 which corresponds to Figure 1. For convenience sake, the origin and terminal point of c_i is denoted by i and $\bar{i}$ respectively. The intersection matrix is as follows [17, pp. 482]

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

We can take many bases of $H_1(X;\mathbb{Z})$. For example, we [15] chose another basis $\{\ell_i\}_{i=1,2,\dots,6}$ such that $\ell_i = c_{i-1} \cdot c_i^{-1}$ for $C_{7,2}$.

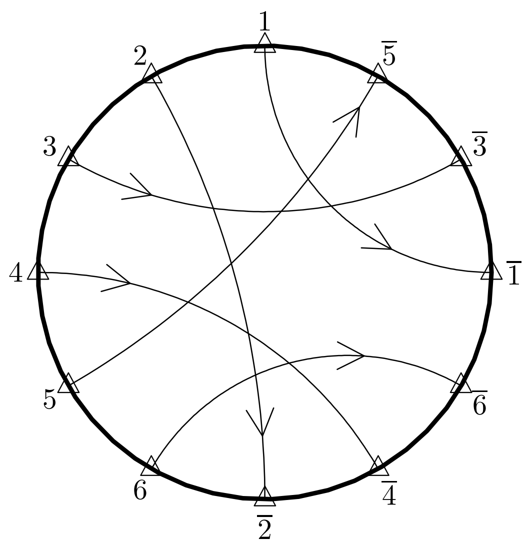

FIGURE 2. An oriented chord diagram of $C_{7,2}$

2.3. **Linear chord diagrams.** Using Bene's method [3], we identify oriented chord diagrams with oriented linear chord diagrams. We define a linear chord diagram in the plane with k chords in the following way. It is the interval $[0, 2k]$, together with k simple arcs in the upper half plane whose endpoints are attached to the interval at the integer points $\{1, 2, \dots, 2k\}$. If the linear chord diagrams are oriented, we call it an oriented linear chord diagram.

Cut open at a certain point on S^1 in an oriented chord diagram in the subsection 2.1. By identifying $2g$ chords with loops $\{c_1, c_2, \dots, c_{2g}\}$, we have the oriented linear chord diagram. Since the intersection number $c_i \cdot c_j$ is independent from the choice of cutting points, we may choose any those points. For $C_{7,2}$, choose a point $\bar{6}$ in Figure 2. Then we have an oriented linear chord diagram in Figure 3.

2.4. **Chord slides.** Bene [3, § 8] used chord slides for the oriented linear chord diagrams. He studied about chord slides with the Whitehead moves on the fatgraphs

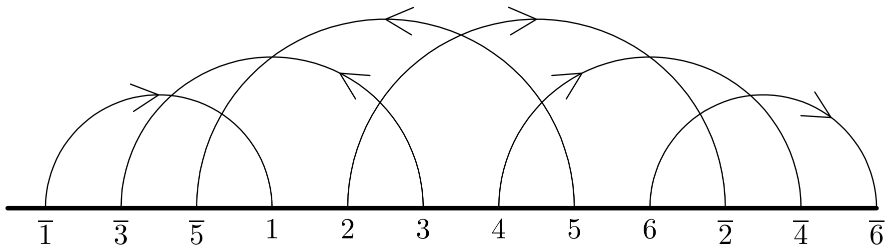

FIGURE 3. An oriented linear chord diagram of $C_{7,2}$

embedded in a surface of genus g with one boundary component. We simply use chord slides to compute the intersection numbers and find the matrix T such that $(c_1, c_2, \ldots, c_{2g})^t T$ is a symplectic basis of $H_1(X; \mathbb{Z})$.

For the oriented linear chord diagram, we define a *chord slide of c_i along c_j for the same position* by the transformation from $(c_1, c_2, \ldots, c_{2g})$ to $(c'_1, c'_2, \ldots, c'_{2g})$ such that

$$c'_k = \begin{cases} c_j + c_i & (k = i) \\ c_k & (k \neq i). \end{cases}$$

as homology classes. For the *opposite position*, the c'_i is replaced with $c_j - c_i$. We define the *position* of a chord slide of c_i along c_j in the linear chord diagram. It is *same position* that the origin(terminal) point of the sliding chord c_i goes to the origin(terminal) point of the slided chord c_j. The others are *opposite position*. We remark the position does not depend on the orientation of chords. See Figure 4 and 5.

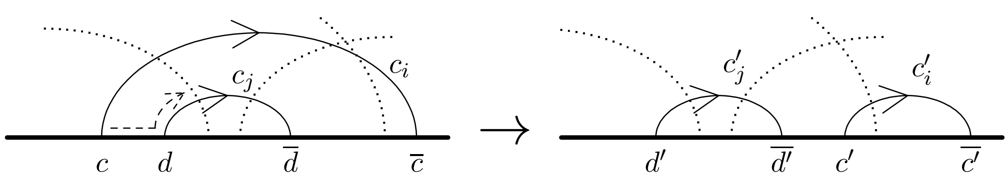

FIGURE 4. A chord slide of c_i along c_j for the same position

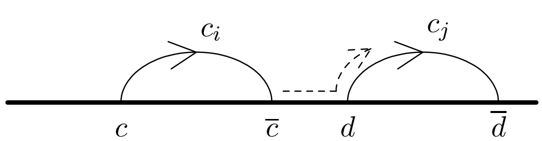

FIGURE 5. A chord slide for the opposite position

For $i \neq j$, let $M_s(\alpha, \beta)$ and $M_o(\alpha, \beta)$ denote the matrices whose (i, j)-th entries are -1 and 1 respectively for $(i, j) = (\alpha, \beta)$ and $\delta_{i,j}$ for $(i, j) \neq (\alpha, \beta)$. Here $\delta_{i,j}$ is Kronecker's delta. After a chord slide of c_i along c_j for the same position, we have

$$(c'_1, c'_2, \ldots, c'_{2g}) = (c_1, c_2, \ldots, c_{2g})^t M_s(i, j).$$

For the opposite position, ${}^t M_s(i, j)$ may be replaced with ${}^t M_o(i, j)$. Let $s_{i,j}(A)$ and $o_{i,j}(A)$ denote the matrices $M_s(i, j) A^t M_s(i, j)$ and $M_o(i, j) A^t M_o(i, j)$ respectively. By seeing the change of the intersection numbers of the chords $(c_1, c_2, \ldots, c_{2g})$, we have

Proposition 2.1. *We consider loops $\{c_k\}_{k=1,2,\dots,2g}$ as chords in the oriented liner chord diagram and A its intersection matrix. If we slide c_i along c_j for the same position, the intersection matrix of $\{c'_k\}_{k=1,2,\dots,2g}$ is $s_{i,j}(A)$. For the opposite position, it is $o_{i,j}(A)$.*

Using this proposition, we have only to deform the intersection matrix into a $2g \times 2g$ symplectic matrix $\begin{pmatrix} O & I_g \\ -I_g & O \end{pmatrix}$. Here I_g is the identity matrix of size g.

For the Klein quartic $C_{7,2}$, we have the matrices $s_{4,2} \circ s_{5,3} \circ o_{5,1} \circ o_{5,3} \circ o_{2,5} \circ s_{5,3}(A)$ and $M = M_s(4,2)M_s(5,3)M_o(5,1)M_o(5,3)M_o(2,5)M_s(5,3)$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

respectively. We denote $(c_1, c_2, \dots, c_6)^t M$ by $(c'_1, c'_2, \dots, c'_6)$. Interchange $\{c'_3, c'_4\}$ and $\{c'_5, c'_6\}$, i.e., two rows $R_3 \leftrightarrow R_4$ and $R_5 \leftrightarrow R_6$ of M. The resulting matrix is denoted by $T_{7,2}$. We have the following matrices $T_{7,2}$ and $T_{7,2}A^tT_{7,2}$ then we obtain a symplectic basis $(a_1, a_2, a_3, b_1, b_2, b_3) = (c_1, c_2, \dots, c_6)^t T_{7,2}$

$$T_{7,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}, \quad T_{7,2}A^tT_{7,2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $T_{7,2}$ is different from one of [17].

2.5. **Period matrices.** We introduce a method for calculating the period matrix of X using holomorphic 1-forms of Bennama and Carbonne [5]. We compute those of $C_{7,1}$ and $C_{7,2}$.

Let $H^{1,0}(X)$ be the space of the holomorphic 1-forms on X. The floor function is denoted by $\lfloor \cdot \rfloor$. For $n = 1, 2, \dots, p-1 (= 2g)$, we define α_l and α_m by $\left\lfloor \frac{nl}{p} \right\rfloor$ and $\left\lfloor \frac{nm}{p} \right\rfloor$ respectively. Set $d_n = \left\lfloor \frac{n(l+m)}{p} \right\rfloor - \alpha_l - \alpha_m - 1$. Bennama and Carbonne [5] obtained that a basis of $H^{1,0}(X)$ is

$$\omega_{n,d} = \frac{x^{\alpha_l}(1-x)^{\alpha_m}x^d}{y^n} \text{ with } 0 \le d \le d_n \text{ and } 1 \le n \le p-1.$$

Take a symplectic basis $\{a_i, b_i\}_{i=1,2,\dots,g}$ of $H_1(X;\mathbb{Z})$, *i.e.*, their intersection numbers are $a_i \cdot b_j = \delta_{i,j}$ and $a_i \cdot a_j = b_i \cdot b_j = 0$. Here $\delta_{i,j}$ is Kronecker's delta. We define two $g \times g$ matrices Ω_A and Ω_B by $\left(\int_{a_j} \omega_i\right)_{i,j}$ and $\left(\int_{a_j} \omega_i\right)_{i,j}$. It is known that the period matrix with respect to $\{a_i, b_i\}_{i=1,2,\dots,g}$ is obtained by $\Omega_A^{-1}\Omega_B$. See [12] for example.

We explain the case $C_{7,1}$ and $C_{7,2}$. For $k = 1, 2$, we define a basis $\omega_1^k, \omega_2^k, \omega_3^k$ of $H^{1,0}(C_{7,k})$ as follows:

	ω_1^k	ω_2^k	ω_3^k
$C_{7,1}$	$\dfrac{dx}{y^6}$	$\dfrac{dx}{y^5}$	$\dfrac{dx}{y^4}$
$C_{7,2}$	$\dfrac{(1-x)dx}{y^6}$	$\dfrac{(1-x)dx}{y^5}$	$\dfrac{dx}{y^3}$

Let $B(u, v)$ be the beta function $\displaystyle\int_0^1 t^{u-1}(1-t)^{v-1}dt$ for $u, v > 0$. Put $(h_1, h_2, h_3, h_4) = (1/7, 2/7, 4/7, 1/7)$. From the equation $\int_{I_0} \omega_i^k dx = B(h_i, h_{i+k-1})$, we have

Lemma 2.2.

$$\int_{c_j} \omega_i^k = B(h_i, h_{i+k-1}) \begin{cases} (1 - \zeta_7^{ij}) & (k = 1), \\ (1 - \zeta_7^{7h_i j}) & (k = 2). \end{cases}$$

Remark 2.3. These integrals depend only on the cohomology class of ω_i^k and the homology class of c_j.

For $k = 1, 2$, let A_k and B_k be the 3×3 matrices $\left(\int_{a_j} \omega_i^k\right)_{i,j}$ and $\left(\int_{b_j} \omega_i^k\right)_{i,j}$ respectively. We have the equation as 3×6 matrices

$$(A_k, B_k) = \left(\int_{c_j} \omega_i^k\right)_{i,j} {}^tT_{7,k}.$$

Here we denote $T_{7,1} = T$ for the case $p = 7$ in Section 3. We have the period matrix of $C_{7,k}$.

Theorem 2.4. *Let $\tau_{7,k}$ be the period matrix $A_k^{-1}B_k$ of $C_{7,k}$. Then we have*

$$\tau_{7,1} = \begin{pmatrix} -\zeta^5 & -2-\zeta^2-\zeta^4-\zeta^5 & \zeta+\zeta^3+\zeta^5 \\ -2-\zeta^2-\zeta^4-\zeta^5 & \zeta+2\zeta^3-\zeta^4+\zeta^5 & 1+\zeta^2+\zeta^3+\zeta^5 \\ \zeta+\zeta^3+\zeta^5 & 1+\zeta^2+\zeta^3+\zeta^5 & \zeta^2 \end{pmatrix}$$

and

$$\tau_{7,2} = \frac{1}{16} \begin{pmatrix} 7+5\sqrt{-7} & 5-\sqrt{-7} & 10-2\sqrt{-7} \\ 5-\sqrt{-7} & 7+5\sqrt{-7} & 6+2\sqrt{-7} \\ 10-2\sqrt{-7} & 6+2\sqrt{-7} & -4+4\sqrt{-7} \end{pmatrix},$$

where $\zeta = \zeta_7$.

We remark that we use the equation $\zeta_7+\zeta_7^2+\zeta_7^4 = (-1+\sqrt{-7})/2$ for the computation of $\tau_{7,2}$.

3. Two symplectic bases of a kind of hyperelliptic curves

For *odd* integer $q \geq 5$, let $C_{q,1}$ be a plane algebraic curve defined by the affine equation $y^q = x(1-x)$. We obtain two symplectic bases of the first integral homology group $H_1(C_{q,1}; \mathbb{Z})$. By substituting $y = \sqrt[q]{\dfrac{1}{4}}\, z$ and $x = \dfrac{\sqrt{-1}\,w - 1}{2}$ into the above equation, we have $w^2 = z^q - 1$. So, it is a hyperelliptic curve of genus $g = (q-1)/2$. Set

the order q holomorphic automorphism $\sigma(x,y)=(x,\zeta y)$ with $\zeta=\zeta_q=\exp(2\pi\sqrt{-1}/q)$. Using these coordinates z and w, we define a loop $\gamma_k\colon [0,1]\to C_{q,1}$, $k=0,1,\ldots,2g$, by

$$\gamma_k(t)=\begin{cases}(\zeta^k\cdot 2t,\sqrt{-1}\sqrt{1-(2t)^p}) & (0\le t\le 1/2),\\ (\zeta^k(2-2t),-\sqrt{-1}\sqrt{1-(2-2t)^p}) & (1/2\le t\le 1).\end{cases}$$

We define the path $I_0\colon [0,1]\to C_{q,1}$ similarly in Subsection 2.1. It is easy to prove

Lemma 3.1. *For $k=0,1,\ldots,2g$, we have two paths $(\sigma_*)^k I_0$ and γ_k are homotopic with relative endpoints.*

We introduce a well-known fact. See [1] for example.

Proposition 3.2. *For $i=1,2,\ldots,g$, we denote $A_i=\gamma_{2i-1}\cdot\gamma_{2i}^{-1}$ and $B_i=\gamma_{2i-1}\cdot\gamma_{2i-2}^{-1}\cdot\cdots\cdot\gamma_1\cdot\gamma_0^{-1}$. Then we have $\{A_i,B_i\}_{i=1,2,\ldots,g}$ is a symplectic basis of $H_1(C_{q,1};\mathbb{Z})$.*

We call this basis a natural type. In fact, this proposition immediately follows from a two-sheeted covering $C_{q,1}\ni(z,w)\to z\in\mathbb{C}P^1$ branched over $2g+2$ points $\{1,\zeta,\zeta^2,\ldots,\zeta^{2g},\infty\}\subset\mathbb{C}P^1$. We find another symplectic basis of $H_1(C_{q,1};\mathbb{Z})$. Although q is not prime in general, the method of the previous section can be applied to $C_{q,1}$ similarly. We recall $c_i=I_0\cdot(\sigma_*)^i I_0^{-1}$. Using the oriented linear chord diagram in Figure 6, we have the the intersection numbers of c_i's

$$c_i\cdot c_j=\begin{cases}1 & (i<j)\\ 0 & (i=j)\\ -1 & (i>j)\end{cases}$$

Lemma 3.3. *Let $M=(c_i\cdot c_j)_{i,j}$ be the intersection matrix of c_i's. Then, the $2g\times 2g$ matrix*

$$s_{2,4}\circ s_{4,6}\circ\cdots\circ s_{2g-2,2g}\circ o_{3,2}\circ o_{4,3}\circ\cdots\circ o_{2g,2g-1}(M)$$

equal to the $2g\times 2g$-matrix

$$\begin{pmatrix} J & & & O\\ & J & & \\ & & \ddots & \\ O & & & J\end{pmatrix},$$

where $J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

Proof. It is enough only to explain from Figure 6 to Figure 8 via Figure 7. In Figure 6, we first take a chord slide c_{2g} along c_{2g-1} for an opposite position. Similarly, we take chord slides c_{2g-1} along $c_{2g-2},\ldots,$ and c_3 along c_2 for opposite positions. We obtain Figure 7. Endpoints series in this figure are

$$1,\overline{3},2,\overline{4},3,\overline{5},4,\overline{6},5,\ldots,\overline{2i+1},2i,\overline{2i+2},2i+1,\ldots,\overline{2g-1},2g-2,\overline{2g},2g-1,1,2.$$

Next we take chord slides c_{2g-2} along c_{2g}, c_{2g-2} along $c_{2g-4},\ldots,$ and c_2 along c_4 for same positions. Finally, we have Figure 8. Endpoints series in this figure are

$$1,\overline{3},4,3,\overline{5},6,5,\ldots,\overline{2i+1},2i,\overline{2i+1},\ldots,\overline{2g-1},\overline{2g},2g-1,2g,2g-2,\ldots,2i,\ldots,2,\overline{1},\overline{2}.$$

We show the intersection matrices M, $o_{3,2}\circ o_{4,3}\circ o_{5,4}\circ o_{6,5}(M)$, and $s_{2,4}\circ s_{4,6}\circ o_{3,2}\circ o_{4,3}\circ o_{5,4}\circ o_{6,5}(M)$ corresponding to Figure 6, 7, and 8 respectively for the case $g=3$.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\text{and} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

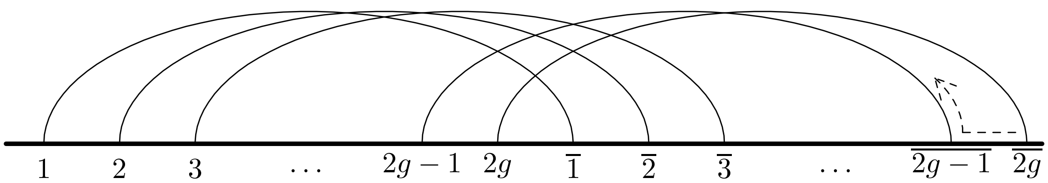

FIGURE 6.

□

Let $c'_1, c'_2, \dots, c'_{2g}$ be a basis of $H_1(C_{q,1}; \mathbb{Z})$ obtained by the above Lemma. We interchange this basis, then we have a symplectic basis

$$(a_1, \dots, a_g, b_1, \dots, b_g) = (c'_1, c'_3, \dots, c'_{2g-1}, c'_2, c'_4, \dots, c'_{2g}).$$

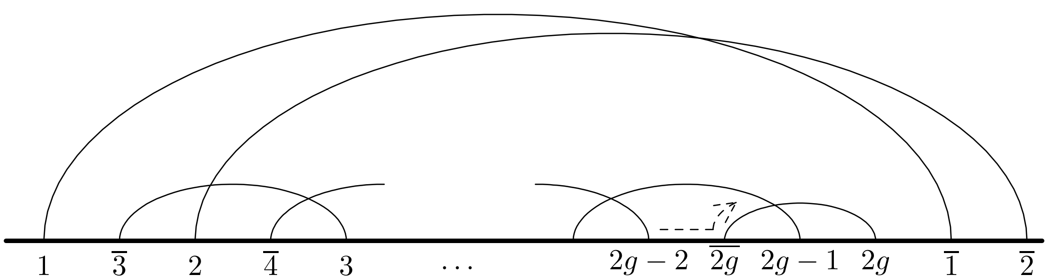

FIGURE 7.

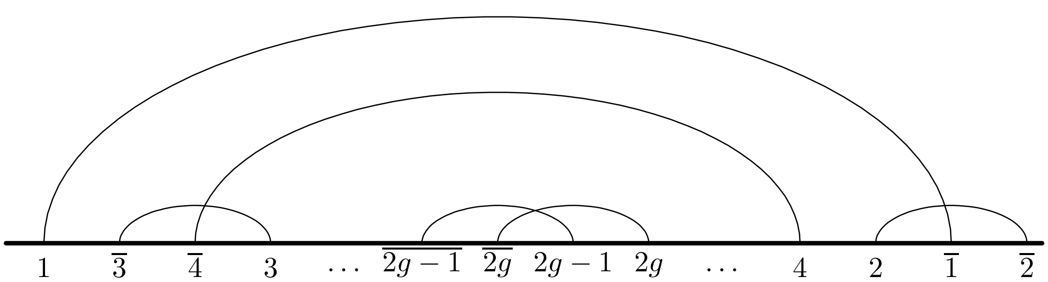

FIGURE 8.

We interchange the raws of the matrix

$$M_s(2,4)M_s(4,6)\cdots M_s(2g-2,2g)M_o(3,2)M_o(4,3)\cdots M_o(2g,2g-1)$$

in the following way. All the odd raws move to $1,2,\dots,g$-th ones and even $g+1,g+2,\dots,2g$. This resulting matrix is denoted by T.

Theorem 3.4. *Set $(a_1,\dots,a_g,b_1,\dots,b_g)=(c_1,c_2,\dots,c_{2g})^tT$. Then, $\{a_i,b_i\}_{i=1}^g$ is a symplectic basis of $H_1(C_{q,1};\mathbb{Z})$.*

Remark 3.5. Let $\vec{t_i}$ be the raw vector component of T for $i=1,2,\dots,2g$. Then we have

$$\vec{t_i}=\begin{cases}(1,0^{2g-1}) & (i=1)\\ (0^{2i-3},-1,1,0^{2(g-i)+1}) & (i=2,3,\dots,g)\\ (0,1,(-1,1)^{g-1}) & (i=g+1)\\ (0^{2(i-1)},(-1,1)^{g-i+1}) & (i=g+2,g+3,\dots,2g)\end{cases}$$

Here we denote $0^n=\underbrace{0,0,\dots,0}_{n}$ and $(-1,1)^m=\underbrace{-1,1,-1,-1,\dots,-1,1}_{m\text{ pairs}}$.

We prove that two symplectic bases $\{A_i,B_i\}_{i=1,2,\dots,g}$ and $\{a_i,b_i\}_{i=1,2,\dots,g}$ are different. From Lemma 3.1, we have the matrix K such that $(A_1,\dots,A_g,B_1,\dots,B_g)=(c_1,c_2,\dots,c_{2g})^tK$

$$K=\begin{pmatrix} -1 & 1 & 0 & & \cdots & & 0\\ 0 & 0 & -1 & 1 & 0 & \cdots & 0\\ & & & \vdots & & & \\ 0 & & \cdots & & 0 & -1 & 1\\ -1 & 0 & & \cdots & & & 0\\ -1 & 1 & -1 & 0 & \cdots & & 0\\ & & & \vdots & & & \\ -1 & 1 & -1 & 1 & \cdots & -1 & 0\end{pmatrix}.$$

The sums of all the components of T and K are equal to $g+1$ and $-g$ respectively. Then we have

Proposition 3.6. *Two symplectic bases $\{A_i,B_i\}_{i=1,2,\dots,g}$ and $\{a_i,b_i\}_{i=1,2,\dots,g}$ are different.*

4. Period matrices of a hyperelliptic curve

We compute the period matrix τ_g of $C_{q,1}$ with respect to a natural symplectic basis $\{A_i,B_i\}_{i=1,2,\dots,g}$ of $H_1(C_{q,1};\mathbb{Z})$. Moreover, we obtain the rerlation between τ_g and Schindler's [13]. Set $\omega_i=\dfrac{dx}{y^{q-i}}$ for $i=1,2,\dots,g$. Bennama [4] proved that $\{\omega_i\}_{i=1,2,\dots,g}$ is a basis of $H^{1,0}(C_{q,1})$. From Subsection 2.5, we have the period of ω_i along c_j

$$\int_{c_j}\omega_i=(1-\zeta^{ij})B(i/q,i/q).$$

For simplicity, we denote $\omega_i'=\omega_i/B(i/q,i/q)$. Two $g\times g$ matrices Ω_A and Ω_B are defined by $\left(\int_{A_j}\omega_i'\right)_{i,j}$ and $\left(\int_{B_j}\omega_i'\right)_{i,j}$ respectively. We have the equation as $g\times 2g$

matrices,

$$(\Omega_A, \Omega_B) = \left(\int_{c_j} \omega_i \right) {}^t K.$$

We have the matrices Ω_A and Ω_B of periods. Tashiro, Yamazaki, Ito, and Higuchi [16] obtained the same result.

Proposition 4.1. *$\Omega_A = \left(\zeta^{i(2j-1)} - \zeta^{2ij}\right)_{i,j}$ and $\Omega_B = \left(\sum_{k=0}^{2j-1} (-1)^{k+1} \zeta^{ik} \right)_{i,j}$.*

Let τ_g denote the period matrix of $C_{q,1}$ with respect to a natural symplectic basis $\{A_i, B_i\}_{i=1,2,\ldots,g}$ of $H_1(C_{q,1}; \mathbb{Z})$. It is known that the matrix τ_g is obtained by $\Omega_A^{-1}\Omega_B$. In order to compute it, we introduce two lemmas. The diagonal matrix whose (i,i)-th entry is a_i denoted by $\operatorname{diag}(a_i)_i$.

Lemma 4.2. *We have*

$$\Omega_A = -\operatorname{diag}\left(-1 + \zeta^i\right)_i \operatorname{diag}\left(\zeta^i\right)_i \left(\zeta^{2i(j-1)}\right)_{i,j} \text{ and}$$

$$\Omega_B = \operatorname{diag}\left(-1 + \zeta^i\right)_i \left(\sum_{k=0}^{j-1} \zeta^{2ik} \right)_{i,j}.$$

Remark 4.3. The matrix $\left(\zeta^{2i(j-1)}\right)_{i,j}$ is called the Vandermonde matrix.

For variables $x_1, x_2, \ldots, x_n$, we denote by $\sigma_i(x_1, x_2, \ldots, x_n)$ the symmetric polynomial

$$\sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} \cdots x_{j_i}$$

for $1 \le i \le n$ and $\sigma_0(x_1, x_2, \ldots, x_n) = 1$. Knuth [11, Excercise 40 in §1.2.3] pointed out the inverse matrix of a Vandermonde matrix.

Lemma 4.4. *Let $a_1, a_2, \ldots, a_n$ be distinct complex constants. We denote a Vandermonde matrix of size n by $V_n = \left(a_i^{j-1}\right)_{i,j}$. Then we have the inverse matrix of V_n is*

$$V_n^{-1} = \left((-1)^{i-1} \frac{\sigma_{n-i}(a_1, \ldots, \widehat{a_j}, \ldots, a_n)}{\prod_{m=1, m\neq j}^{n} (a_m - a_j)} \right)_{i,j},$$

where the 'hat' symbol $\widehat{\ }$ over a_j indicates that this element is deleted from the sequence $a_1, \ldots, a_n$.

From the above two lemmas, we have.

Theorem 4.5. *We have the period matrix τ_g of $C_{q,1}$ with respect to the symplectic basis $\{A_i, B_i\}_{i=1,2,\ldots,g}$*

$$\tau_g = \left(\sum_{k=1}^{g} \frac{(-1)^{i+g}}{2g+1} (1 - \zeta^{2kj}) \sigma_{g-i}(\zeta^2, \zeta^4, \ldots, \widehat{\zeta^{2j}}, \ldots, \zeta^{2g}) \prod_{m=g-k+1}^{2g-k} (1 - \zeta^{2m}) \right)_{i,j}.$$

Proof. We compute $\Omega_A^{-1}\Omega_B$ as follows

$$\begin{aligned}\Omega_A^{-1}\Omega_B &= -\left(\zeta^{2i(j-1)}\right)_{i,j}^{-1}\operatorname{diag}\left(\zeta^{-i}\right)_i\left(\sum_{k=0}^{j-1}\zeta^{2ik}\right)_{i,j}\\ &= -\left(\zeta^{2i(j-1)}\right)_{i,j}^{-1}\left(\zeta^{-i}\frac{1-\zeta^{2ij}}{1-\zeta^{2i}}\right)_{i,j}.\end{aligned}$$

From the equation

$$\left(\zeta^{2i(j-1)}\right)_{i,j}^{-1} = \left((-1)^{i-1}\frac{\sigma_{g-i}(\zeta^2,\zeta^4,\dots,\widehat{\zeta^{2j}},\dots,\zeta^{2g})}{\prod_{m=1,m\neq j}^{n}(\zeta^{2m}-\zeta^{2k})}\right)_{i,j},$$

we have (i,j)-th entry of τ_g

$$\sum_{k=1}^{g}(-1)^i\frac{\sigma_{g-i}(\zeta^2,\zeta^4,\dots,\widehat{\zeta^{2j}},\dots,\zeta^{2g})}{\prod_{m=1,m\neq k}^{g}(\zeta^{2m}-\zeta^{2k})}\frac{\zeta^{-k}(1-\zeta^{2kj})}{1-\zeta^{2k}}.$$

Using the following lemma, we obtain the theorem. □

Lemma 4.6. *With the notation as above, we have the equation*

$$\frac{1}{\zeta^k(1-\zeta^{2k})\prod_{m=1,m\neq k}^{g}(\zeta^{2m}-\zeta^{2k})} = \frac{(-1)^g}{2g+1}\prod_{m=g-k+1}^{2g-k}(1-\zeta^{2m}),$$

for each $k = 1, 2, \dots, g$.

Proof. We compute the denominator of the LHS

$$\begin{aligned}\zeta^k(1-\zeta^{2k})\prod_{m=1,m\neq k}^{g}(\zeta^{2m}-\zeta^{2k}) &= \zeta^{k+2k(g-1)+2k}(\zeta^{-2k}-1)\prod_{m=1,m\neq k}^{g}(\zeta^{2(m-k)}-1)\\ &= (-1)^g\prod_{m=-k,m\neq 0}^{g-k}(1-\zeta^{2m}).\end{aligned}$$

The result immediately follows from this and the equation

$$\prod_{m=-k,m\neq 0}^{g-k}(1-\zeta^{2m})\prod_{m=g-k+1}^{2g-k}(1-\zeta^{2m}) = \prod_{l=1}^{2g}(1-\zeta^l) = 2g+1.$$

□

Set $\zeta = \zeta_7$. We calculate τ_3 of $C_{7,1}$

$$\begin{pmatrix} -\zeta^5 & -1-\zeta^2-\zeta^4-\zeta^5 & 1+\zeta+\zeta^3+\zeta^5\\ -1-\zeta^2-\zeta^4-\zeta^5 & 1+\zeta+2\zeta^3-\zeta^4+\zeta^5 & 2+\zeta^2+\zeta^3+\zeta^5\\ 1+\zeta+\zeta^3+\zeta^5 & 2+\zeta^2+\zeta^3+\zeta^5 & 1+\zeta^2 \end{pmatrix}.$$

In general, the period matrix depends only on the choice of a symplectic basis and the complex structure of a compact Riemann surface. Two period matrices τ_g and τ_g' obtained from the same compact Riemann surface if and only if there exists a symplectic matrix $\begin{pmatrix} P & Q\\ R & S\end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$ such that $\tau_g' = (P+\tau_g R)^{-1}(Q+\tau_g S)$. Here P, Q, R, and S are $g\times g$ $\mathbb{Z}$-coefficient matrices.

The symplectic matrix $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ for two period matrices of $C_{q,1}$ with respect to $\{a_i, b_i\}_{i=1,2,\dots,g}$ and $\{A_i, B_i\}_{i=1,2,\dots,g}$ is given by ${}^tK({}^tT)^{-1}$. We call it H. For $g = 3$, this matrix can be computed as

$$H = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix} \in \mathrm{Sp}(6, \mathbb{Z}).$$

We have $\tau_{7,1} = (P + \tau_3 R)^{-1}(Q + \tau_3 S)$. The matrix H can be obtained similarly for $g > 3$. But the period matrix with respect to a symplectic basis $\{a_i, b_i\}_{i=1,2,\dots,g}$ is complicated.

We introduce Schindler's period matrix, denoted by τ_g^S, for the hyperelliptic curve defined by the affine equation $w_1^2 = z_1(z_1^{2g+1} - 1)$. It is denoted by $C'_{q,1}$ This is biholomorphic to $C_{q,1}$. For $i = 1, 2, \dots, g$, elements t_i of the q-th cyclotomic field $\mathbb{Q}(\zeta)$ are defined as follows:

$$\begin{aligned} t_1 &= (-1)^g \zeta^{g^2} && (i = 1), \\ t_2 &= t_1\left(1 - \frac{1}{1+\zeta}\right) && (i = 2), \\ t_{i+1} &= \frac{t_1(1 - \sum_{k=2}^{i} \zeta^{g-i+k-1} t_k t_{i-k+2})}{1 + \zeta^{-i}} && (i = 2, 3, \dots, g-1). \end{aligned}$$

Theorem 4.7 (Schindler [13])**.** *The (i, j)-th entry of the period matrix τ_g^S is obtained by*

$$s_{i,j} = 1 - \frac{1}{t_1} \sum_{k=1}^{i} t_k t_{j-i+k}$$

for $1 \le i \le j \le g$ and $s_{j,i}$ for $g \ge i > j \ge 1$.

If we set $z_1 = 1/z$ and $w_1 = \sqrt{-1}\, w/z^{g+1}$, we obtain the biholomorphism from $C'_{q,1}$ to $C_{q,1}$. This implies that a symplectic basis of Schindler [13] is given by

$$(A_g, A_{g-1}, \dots, A_1, B_g, B_{g-1}, \dots, B_1) = (A_1, A_2, \dots, A_g, B_1, B_2, \dots, B_g) \begin{pmatrix} L_g & O \\ O & L_g \end{pmatrix},$$

using the symplectic basis of natural type. Here the (i, j)-th entry of the $g \times g$ matrix L_g is 1 for $i + j = g + 1$ and 0 otherwise. It immediately follows that $L_g^{-1} = L_g$ and $\begin{pmatrix} L_g & O \\ O & L_g \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$. Then we have

Proposition 4.8. *The relation between two period matrices τ_g and τ_g^S is obtained by*

$$\tau_g^S = L_g \tau_g L_g.$$

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